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LYAPUNOV FUNCTIONS FOR INVESTIGATING THE GLOBAL STABILITY OF NON-LINEAR SYSTEMS*

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The sufficient conditions of the sign determinacy of a sum of polylinear forms are obtained. From these systems the Lyapunov functions which are used to derive the sufficient conditions of the global asymptotic stability of the non-perturbed motion of non-linear systems are set up. At the same time the perturbed motion is described by a set of ordinary differential equations with the right-hand side in the form of the sum of homogeneous polynomials. An application to the analysis of the stability of winged aircraft is considered.

1. We shall first obtain the conditions of sign determinacy of a sum of the forms

$$F(\mathbf{x}) = \sum_{s=2k}^{2m} X^{(s)}(\mathbf{x}, A_{i_1\ldots i_s}), \quad \mathbf{x} = (x_1, \ldots, x_n) \in R^n \tag{1.1}$$

where

$$X^{(s)}(\mathbf{x}, A_{i_1\ldots i_s}) = \sum_{i_1=1}^{n} \cdots \sum_{i_s=i_{s-1}}^{n} A_{i_1\ldots i_s} x_{i_1} \cdots x_{i_s} \tag{1.2}$$
$$(1 \leqslant i_1 \leqslant \ldots \leqslant i_s \leqslant n)$$

is a polylinear form of degree $s = 2k, 2k+1, \ldots, 2m$; $A_{i_1\ldots i_s}$ are real numbers, k, m, s, n are positive integers $(k \leqslant m)$. Similar terms of the form (1.2) are derived. In form (1.2) the terms are ordered lexicographically.

In the sum of forms (1.1) the forms follow each other in order of increasing or decreasing degree of these forms. However, in any case the degree of the first and last forms in this sum (1.1) is even, i.e. $2k$ and $2m$. We will call these functions (1.1) sums of forms, using the bordered forms of even degree.

It is required to find conditions connecting the coefficients $A_{i_1\ldots i_s}$, for which the sum of forms (1.1) will be positive definite:

$$F(\mathbf{x}) > 0, \ \forall \mathbf{x} \neq 0; \quad F(0) = 0 \tag{1.3}$$

To solve the problem we will introduce the mapping

$$\begin{gathered}
y_1 = x_1^m, \quad y_2 = x_1^{m-1}x_2, \quad y_3 = x_1^{m-1}x_3, \ldots, \quad y_n = x_1^{m-1}x_n \\
y_{n+1} = x_1^{m-2}x_2^2, \quad y_{n+2} = x_1^{m-2}x_2x_3, \quad y_{n+3} = x_1^{m-2}x_2x_4, \ldots \\
\ldots, y_j = x_{i_1}x_{i_2}\ldots x_{i_r}, \ldots, y_{N-1} = x_{n-1}x_n^{k-1}, \quad y_N = x_n^k \\
j = 1, \ldots, N; \quad i_1, \ldots, i_r = 1, \ldots, n; \quad i_1 \leqslant \ldots \leqslant i_r, \\
j \rightleftarrows i_1 \ldots i_r, \quad m \geqslant r \geqslant k
\end{gathered} \tag{1.4}$$

The sequence of values taken by the index j, and the corresponding values taken by the group index $i_1i_2\ldots i_r$ can be represented in strictly algorithmic form. Initially the values of the index j increase from 1 to N, for $r = m$, for which the group index $i_1i_2\ldots i_r$ changes, beginning from $\underbrace{11\ldots11}_{m}$ with respect to the lexicographically arranged sequence. Then the process by which the index j increases continues for $r = m-1$ etc., to $r = k$. Thus, between the index j and the group index $i_1i_2\ldots i_r$ there is the one-to-one correspondence:

$$\begin{gathered}
1 \rightleftarrows \underbrace{11\ldots11}_{m}, \quad 2 \rightleftarrows \underbrace{11\ldots1}_{m-1}2, \quad 3 \rightleftarrows \underbrace{11\ldots1}_{m-1}3, \ldots, \quad n \rightleftarrows \underbrace{11\ldots1}_{m-1}n \\
n+1 \rightleftarrows \underbrace{11\ldots1}_{m-2}22, \quad n+2 \rightleftarrows \underbrace{11\ldots1}_{}23, \\
N-1 \rightleftarrows (n-1)\underbrace{n\ldots nn}_{k-1}, \quad N \rightleftarrows \underbrace{nn\ldots nn}_{k}
\end{gathered}$$

The overall number N of functions y_j occurring in the mapping (1.4) is calculated using Eq./1/

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$$N = \sum_{r=k}^{m} N_r, \quad N_r = C^r_{n+r-1} \tag{1.5}$$

where N_r is the number of terms of the form of degree r, and C^r_{n+r+1} is the number of combinations of $n+r-1$ taken r at a time.

We shall formulate $\mathbf{y} = (y_1, \ldots, y_N)$ and shall use an abbreviated notation of the mapping (1.4): $\mathbf{y} = \mathbf{y}(\mathbf{x})$. It is well-known that the vectors $\mathbf{y} = (y_1, \ldots, y_N)$ form the linear space R^N /2/. We shall introduce the quadratic form in the space

$$P(\mathbf{y}) = \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} B_{j_1j_2} y_{j_1} y_{j_2}, \quad B_{j_1j_2} = B_{j_2j_1} \tag{1.6}$$

We shall substitute the values $y_1, \ldots, y_N$ (1.4) into the function (1.6) and reduce similar terms. In order not to lose the connection between the indices j_1, j_2 and the group indices $i_1, \ldots i_r, i_{r+1} \ldots i_s$ respectively, we shall denote the coefficients $B_{j_1j_2}$ in the following way:

$$\begin{gathered} B_{j_1j_2} = B^*_{i_1\ldots i_r,\, i_{r+1}\ldots i_s} \\ j_1, j_2 = 1, \ldots, N; \quad i_1, \ldots, i_s = 1, \ldots, n; \quad j_1 \rightleftarrows i_1 \ldots i_r, \\ j_2 \rightleftarrows i_{r+1} \ldots i_s \end{gathered} \tag{1.7}$$

where the indices i_r and i_{r+1} are not connected by inequalities, as occurs for the indices:

$$i_1 \leqslant i_2 \leqslant \ldots \leqslant i_r, \quad i_{r+1} \leqslant i_{r+2} \leqslant \ldots \leqslant i_s \tag{1.8}$$

Identically equating the right-hand sides of the equation obtained for $P(\mathbf{y}(\mathbf{x}))$ (1.6) and for $F(\mathbf{x})$ (1.1), we will obtain - bearing in mind the notation (1.7) - the connection between the coefficients of the functions (1.1) and (1.6) in the form

$$\begin{gathered} A_{i_1\ldots i_s} = (2-\delta_{j_1j_2}) \sum_{r=k}^{m} \sum\nolimits^* B_{j_1j_2} = (2-\delta_{j_1j_2}) \sum_{r=k}^{m} \sum\nolimits^* B^*_{i_1\ldots i_r,\, i_{r+1}\ldots i_s} \\ j_1, j_2 = 1, \ldots, N; \; i_1, \ldots, i_s = 1, \ldots, n; \; s = 2k, \\ 2k+1, \ldots, 2m \end{gathered} \tag{1.9}$$

where $\delta_{j_1j_2}$ is the Kronecker delta, and Σ^* is the symbol of summation with respect to the permutations of the integral values of the indices $i_1, \ldots, i_s$, for which conditions (1.8) are satisfied.

When setting up the specific Eq.(1.9) it is convenient, in practice, to specify first the values of the indices $i_1, \ldots, i_s$ and further, permuting the integral values of the indices $i_1, \ldots, i_s$ while preserving conditions (1.8), to write the similar terms with $B_{j_1j_2}$ $(j_1 \rightleftarrows i_1 \ldots i_r,$ $j_2 \rightleftarrows i_{r+1} \ldots i_s)$, corresponding to the coefficient $A_{i_1\ldots i_s}$.

Thus, when Eqs.(1.4) and (1.9) hold $\forall \mathbf{x} \in R^n$, $\exists \mathbf{y} \in G_y^*$, such that

$$F(\mathbf{x}) = P(\mathbf{y}(\mathbf{x})) \tag{1.10}$$

where G_y^* is the range of values of the mapping (1.4) in R^N $(G_y^* \subset R^N)$, whilst the point $\mathbf{y} = 0 \in G_y^*$. Therefore, property A /3/ holds for $F(\mathbf{x})$ (1.1), $P(\mathbf{y})$ (1.6) and the mapping (1.4). At the same time property B /3/ also holds, i.e. $\forall x_j \neq 0$, $\exists y_i = x_j^r \neq 0$, $r = k, k+1, \ldots, m$.

When properties A and B hold for the positive definiteness of the sum of forms $F(\mathbf{x})$ (1.1) according to Theorem 1 /3/ it is sufficient that there is a positive definite quadratic form $P(\mathbf{y})$ (1.6). Noting the positive definite quadratic form criterion (1.6) and expressing the coefficients $B_{j_1j_2}$ in terms of the coefficients $A_{i_1\ldots i_s}$ from Eqs.(1.9) we will obtain the conditions of sign determinacy of the sum of the forms (1.1). In particular, we can use Sylvester's criterion /4/ or the criterion obtained in /3/, which has a recurrent form and is simple in calculation respects.

2. We shall find the positive definite conditions of the sum of the forms with constant real coefficients

$$\begin{aligned} F(\mathbf{x}) = {} & A_{1111}x_1^4 + A_{1112}x_1^3x_2 + A_{1122}x_1^2x_2^2 + A_{1222}x_1x_2^3 + \\ & A_{2222}x_2^4 + A_{111}x_1^3 + A_{112}x_1^2x_2 + A_{122}x_1x_2^2 + A_{222}x_2^3 + \\ & A_{11}x_1^2 + A_{12}x_1x_2 + A_{22}x_2^2 \end{aligned} \tag{2.1}$$

Mapping (1.4) has the form

$$\begin{gathered} y_1 = x_1^2,\; y_2 = x_1x_2,\; y_3 = x_2^2,\; y_4 = x_1,\; y_5 = x_2 \\ r = 1, 2;\; N_1 = 2,\; N_2 = 3,\; N = N_1 + N_2 = 5 \end{gathered} \tag{2.2}$$

We shall write the quadratic form in the space of the vectors $\mathbf{y} = (y_1, \ldots, y_5)$

$$P(\mathbf{y}) = \sum_{j_1=1}^{5} \sum_{j_2=1}^{5} B_{j_1j_2} y_{j_1} y_{j_2}, \quad B_{j_1j_2} = B_{j_2j_1} \tag{2.3}$$

We shall substitute the values $y_1, \ldots, y_5$ (2.2) into function (2.3) and shall reduce similar terms. Identically equating the right-hand sides of the expression obtained for $P(y(x))$ (2.3) and the specified function (2.1), we will obtain equations of the form (1.9). Solving them, we arrive at the coefficients of the quadratic form (2.3)

$$
\begin{aligned}
&B_{11} = A_{1111}, \quad B_{12} = \frac{1}{2} A_{1112}, \quad B_{22} = A_{1122} - 2B_{13}, \\
&B_{23} = \frac{1}{2} A_{1222} \\
&B_{33} = A_{2222}, \quad B_{14} = \frac{1}{2} A_{111}, \quad B_{24} = \frac{1}{2} A_{112} - B_{15}, \\
&B_{25} = \frac{1}{2} A_{122} - B_{34} \\
&B_{35} = \frac{1}{2} A_{222}, \quad B_{44} = A_{11}, \quad B_{45} = \frac{1}{2} A_{12}, \quad B_{55} = A_{22}
\end{aligned}
\tag{2.4}
$$

where the real values B_{13}, B_{15}, B_{34} are free and are used to guarantee the positive definiteness of the quadratic form (2.3).

For the positive definiteness of the quadratic form (2.3) according to the criterion of /3/ it is necessary and sufficient that the following real numbers occur:

$$
b_{ij} = \frac{1}{b_{ii}} \Big(B_{ij} - \sum_{k=1}^{i-1} b_{ki} b_{kj}\Big)
\tag{2.5}
$$

$$
i = 1, \ldots, 5; \quad j = i, i+1, \ldots, 5; \ 1 \leqslant k < i
$$

satisfying the condition

$$
b_{ii} \neq 0, \quad \forall i = 1, \ldots, 5
\tag{2.6}
$$

Substituting the values B_{ij} (2.4) into the recurrence formula (2.5), we will obtain the sufficient conditions for the positive definiteness of the sum of the forms (2.1) in the form of the existence of the numbers b_{ij} (2.5), (2.6).

Bearing in mind that for the sign determinacy of $F(x)$ (2.1) the sign determinacy of $P(y(x))$ with respect to (x_1, x_2) is sufficient, we will show that we can relax condition (2.6). Indeed, if the numbers b_{ij}, which satisfy (2.5), (2.6) exist, then the quadratic form (2.3) is positive definite and is represented in the form of the sum of the independent squares /4/

$$
P(y) = \sum_{i=1}^{5} \Big(\sum_{j=1}^{5} b_{ij} y_j\Big)^2
\tag{2.7}
$$

which can be checked by substituting b_{ij} (2.5) into the quadratic form (2.7). Substituting the values $y_1, \ldots, y_5$ (2.2) into function (2.7) and bearing in mind Eq.(1.10), we will obtain a representation of the specified sum of the forms (2.1) in the form

$$
\begin{aligned}
F(x) = {} & (b_{11}x_1^2 + b_{12}x_1x_2 + b_{13}x_2^2 + b_{14}x_1 + b_{15}x_2)^2 + \\
& (b_{22}x_1x_2 + b_{23}x_2^2 + b_{24}x_1 + b_{25}x_2)^2 + \\
& (b_{33}x_2^2 + b_{34}x_1 + b_{35}x_2)^2 + (b_{44}x_1 + b_{45}x_2)^2 + (b_{55}x_2)^2
\end{aligned}
\tag{2.8}
$$

The last two squares of the linear forms $(r = 1)$ in the sum (2.8) form a positive definite quadratic form with respect to (x_1, x_2) and guarantee the positive definiteness of the function $F(x)$ (2.8) under the condition

$$
b_{44} \neq 0, \quad b_{55} \neq 0
\tag{2.9}
$$

which is a weakening of conditions (2.6).

Thus, for the positive definiteness of the specified sum of the forms $F(x)$ (2.1) it is sufficient that the quadratic form $P(y)$ (2.7) is non-negative with respect to $y_1, \ldots, y_5$ and positive definite with respect to the variables $y_4 = x_1$, $y_5 = x_2$ (2.2), which correspond to the quadratic form (the form of degree $2r$ when $r = 1$) in the function $F(x)$ (2.1).

On the other hand, changing the numbering of the variables of the mapping (2.2) in the following way:

$$
y_1 = x_1, \ y_2 = x_2, \ y_3 = x_1^2, \ y_4 = x_1x_2, \ y_5 = x_2^2
\tag{2.10}
$$

and substituting $y_1, \ldots, y_5$ of (2.10) into (2.7), we will obtain, bearing in mind Eq.(1.10),

$$
\begin{aligned}
F(x) = {} & (b_{11}'x_1 + b_{12}'x_2 + b_{13}'x_1^2 + b_{14}'x_1x_2 + b_{15}'x_2^2)^2 + \\
& (b_{22}'x_2 + b_{23}'x_1^2 + b_{24}'x_1x_2 + b_{25}'x_2^2)^2 + \\
& (b_{33}'x_1^2 + b_{34}'x_1x_2 + b_{35}'x_2^2)^2 + (b_{44}'x_1x_2 + b_{45}'x_2^2)^2 + (b_{55}'x_2^2)^2
\end{aligned}
\tag{2.11}
$$

The last three squares of the quadratic form $(r = 2)$ in the sum (2.11) form a positive-definite form of the fourth degree under the condition

$$
b_{33}' \neq 0, \quad b_{44}' \neq 0, \quad b_{55}' \neq 0
\tag{2.12}
$$

and guarantee the positive definiteness of the function $F(\mathbf{x})$ (2.11) /3/. Conditions (2.12) are less rigid than condition (2.6).

Thus, for the positive definiteness of the specified sum of the forms $F(\mathbf{x})$ (2.1) it is sufficient that the quadratic form $P(\mathbf{y})$ (2.7) is non-negative with respect to the variables $y_1, \ldots, y_5$ and positive definite with respect to the variables $y_3 = x_1^2$, $y_4 = x_1x_2$, $y_5 = x_2^2$ (2.10), which correspond to the form of the fourth degree (of degree $2r$ when $r = 2$) in the function $F(\mathbf{x})$ (2.1).

3. Generalizing the conclusion of Sect.2, we will prove the theorem of the sign determinacy of the sum of the forms $F(\mathbf{x})$ (1.1). We will also obtain the recurrent form of the sign determinacy criterion of the quadratic form $P(\mathbf{y})$ (1.5) with respect to part of the variables /5/, i.e. the necessary and sufficient condition for the quadratic form $P(\mathbf{y})$ (1.6) to be sign determinate with respect to all the variables $y_1, \ldots, y_N$ and sign determinate equally with respect to N_r variables $y_{N-N_r+1}, \ldots, y_N$. Note that N_r equals the number of variables of the quadratic form $P(\mathbf{y})$ (1.6) to which, according to Eq.(1.10), the $2r$-degree form in the specified sum of the forms $F(\mathbf{x})$ (1.1), where $r \in \{k, k+1, \ldots, m\}$, corresponds.

Theorem 1. For the positive definiteness of the sum of the forms $F(\mathbf{x})$ (1.1), bordered by the forms of the fourth degree $2k$ and $2m$ $(k \leqslant m)$, it is sufficient that for the mapping (1.4) and Eq.(1.10) the quadratic form $P(\mathbf{y})$ (1.6) is non-negative with respect to the variables $y_1, \ldots, y_N$ and positive definite with respect to N_r variables $y_{N-N_r+1}, \ldots, y_N$, where r takes some value from the set $\{k, k+1, \ldots, m\}$.

Proof. Suppose the conditions of the theorem hold, i.e. the quadratic form $P(\mathbf{y}) > 0$ and is positive definite with respect to N_r variables y_j, which are chosen using the last $j = N - N_r + 1, \ldots, N$. Then using one of the well-known methods /4/ we shall transform the quadratic form $P(\mathbf{y})$ (1.6) into the following sum of squares:

$$P(\mathbf{y}) = \sum_{i=1}^{N-N_r} \Big(\sum_{j=1}^{N} b_{ij}y_j\Big)^2 + \sum_{i=N-N_r+1}^{N} \Big(\sum_{j=i}^{N} b_{ij}y_j\Big)^2 \tag{3.1}$$

$$b_{ii} \neq 0, \quad \forall i = N - N_r + 1, \ldots, N;\ r \in \{k, k+1, \ldots, m\} \tag{3.2}$$

Substituting the values $y_1, \ldots, y_N$ into the function (3.1) and bearing in mind Eq.(1.10), we will obtain a representation of the specified sum of the forms (1.1) in the form of a non-negative part and positive definite form of degree $2r$ /3/. The theorem is proved.

Theorem 2. For the non-negativity of the quadratic form $P(\mathbf{y})$ (1.6) with constant real coefficients $B_{j_1j_2}$ $(j_1, j_2 = 1, \ldots, N)$ with respect to N variables $y_1, \ldots, y_N$ and its positive definiteness equally with respect to N_r variables $y_{N-N_r+1}, \ldots, y_N$ it is necessary and sufficient that we have the real numbers b_{ij}, which are determined by the coefficients $B_{j_1j_2}$ using the recurrence formula

$$b_{ij} = \frac{1}{b_{ii}}\Big(B_{ij} - \sum_{k=1}^{i-1} b_{ki}b_{kj}\Big) \tag{3.3}$$

$$i = 1,\ N - N_r + 1,\ N - N_r + 2, \ldots, N;\ j = i, i+1, \ldots, N;\ 1 \leqslant k < i$$

under the condition

$$b_{ii} \neq 0, \quad \forall i = N - N_r + 1, \ldots, N \tag{3.4}$$

Proof. Necessity. Suppose the quadratic form $P(\mathbf{y})$ (1.6) which is non-negative with respect to N variables $y_1, \ldots, y_N$ and positive definite with respect to N_r variables $y_{N-N_r+1}, \ldots, y_N$ is given. Uisng one of the well-known methods /4/ we shall transform it to a sum of the squares (3.1) under condition (3.2). In representation (3.1) we can select the numbers b_{1j} $(j = 1, \ldots, N)$, such that the following equations hold:

$$b_{ij} = 0, \ \forall i = 2, 3, \ldots, N - N_r;\ j = 1, \ldots, N$$

Indeed, in the matrix

$$\begin{Vmatrix} b_{11} & b_{12} & \cdots & b_{1,N-N_r+1} & \cdots & b_{1N} \\ b_{21} & b_{22} & \cdots & b_{2,N-N_r+1} & \cdots & b_{2N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{N-N_r,1} & b_{N-N_r,2} & \cdots & b_{N-N_r,N-N_r+1} & \cdots & b_{N-N_r,N} \\ 0 & 0 & \cdots & b_{N-N_r+1,N-N_r+1} & \cdots & b_{N-N_r+1,N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \cdots & b_{NN} \end{Vmatrix}$$

of the coefficients b_{ij} of function (3.1) the first $N-N_r$ rows are linearly dependent, since the specified quadratic form $P(\mathbf{y})$ is only sign determinate with respect to the last N_r variables. Therefore the elements of each i-th row $(i=2,3,\ldots,N-N_r)$ differ from the elements of the first row $(i=1)$ arranged in that column by the constant muliplier α_i $(i=2,3,\ldots,N-N_r)$. Consequently

$$\sum_{j=1}^{N} b_{ij}y_j=\alpha_i\sum_{j=1}^{N} b_{1j}y_j \quad (i=2,3,\ldots,N-N_r)$$

In the first $N-N_r$ terms of the sum (3.1) we shall take α_i^2 out of the brackets and reduce similar terms. We will obtain

$$P(\mathbf{y})=\Big(\sum_{j=1}^{N} b_{1j}'y_j\Big)^2+\sum_{i=N-N_r+1}^{N}\Big(\sum_{j=i}^{N} b_{ij}y_j\Big)^2 \tag{3.5}$$
$$b_{1j}'=b_{1j}\Big(\sum_{i=1}^{N-N_r}\alpha_i^2\Big)^{1/2},\quad \alpha_1=1;\quad j=1,\ldots,N$$

Equating the quadratic form (1.6) and (3.5) and comparing the coefficients of identical terms, we obtain the recurrence formula (3.3) under condition (3.4) apart from the notation.

Sufficiency. Suppose the numbers b_{ij}, shown in Theorem 2, exist. Then the specified quadratic form is represented in the form (3.5) under condition (3.4), whence follows the non-negativity of the quadratic form $P(\mathbf{y})$ with respect to N variables $y_1,\ldots,y_N$ and the positive definiteness equally with respect to N_r variables $y_{N-N_r+1},\ldots,y_N$. Theorem 2 is proved.

4. We shall use this result to derive the sufficient conditions of global asymptotic stability of the zero solution of a set of ordinary differential equations with its right-hand side in the form of a sum of homogeneous polynomials

$$\frac{dx_\beta}{dt}=\sum_{s=2h-1}^{2l-1}X_\beta^{(s)}(\mathbf{x},a_{\beta i_1\ldots i_s}),\quad \beta=1,\ldots,n;\quad \mathbf{x}\in R^n \tag{4.1}$$

Here $X_\beta^{(s)}(\mathbf{x},a_{\beta i_1\ldots i_s})$ is a polylinear form of degrees of the form (1.2) with constant real coefficients, $a_{\beta i_1\ldots i_s}(i_1,\ldots,i_s=1,\ldots,n)$ are positive integers $(1\leqslant h<l)$. In the forms of the right-hand side of Eq.(4.1) similar terms are presented and arranged in lexicographic order.

To solve the problem we will use the second Lyapunov method and, in particular, the Barbashin-Krasovskii theorem on global asymptotic stability /6/. We shall seek Lyapunov's function in the set of negative definite functions

$$v(\mathbf{x})=-\frac{1}{2}\sum_{\alpha=1}^{N}\Big(\sum_{r=k}^{m}X_\alpha^{(r)}(\mathbf{x},c_{\alpha i_1\ldots i_r})\Big)^2 \tag{4.2}$$

where $X_\alpha^{(r)}(\mathbf{x},c_{\alpha i_1\ldots i_r})$ is a polylinear form of degree r of the form (1.2) with the constant real coefficients $c_{\alpha i_1\ldots i_r}$ $(\alpha=1,\ldots,N;\ i_1,\ldots,i_r=1,\ldots,n)$, which form an upper triangular $(N\times N)$-matrix when $k\leqslant r\leqslant m$

$$\begin{Vmatrix} c_{11\ldots11} & c_{11\ldots12} & \cdots & c_{1n\ldots nn}\\ 0 & c_{21\ldots12} & \cdots & c_{2n\ldots nn}\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & c_{Nn\ldots nn}\end{Vmatrix} \tag{4.3}$$

in which the last r diagonal coefficients do not equal zero, which, according to Theorem 1, guarantees the sign determinacy of function (4.2); N is the number of terms of the sum of the forms of degree r, where $r=k,k+1,\ldots,m$. The number N is determined using Eq.(1.5).

The function $v(\mathbf{x})$ (4.2) is a sum of the forms, bordered by forms of even degree $2k$ and $2m$. Consequently, the partial derivative of the function $v(\mathbf{x})$ (4.2) with respect to the coordinate x_β $(\beta=1,\ldots,n)$ is a sum of the forms, bordered by the forms of uneven degree $2k-1$ and $2m-1$. The total derivative of the function $v(\mathbf{x})$ (4.2) with respect to time t by virtue of system (4.1) has the form

$$\frac{dv}{dt}=\sum_{\beta=1}^{n}\frac{\partial v}{\partial x_\beta}\frac{dx_\beta}{dt}=\sum_{p=2(h+k-1)}^{2(l+m-1)}X^{(p)}(\mathbf{x},A_{i_1\ldots i_p}) \tag{4.4}$$

where $X^{(p)}(\mathbf{x},A_{i_1\ldots i_p})$ is a polylinear form of degree p of the form (1.2) with constant real coefficients $A_{i_1\ldots i_p}$, which are expressed by the coefficients of system (4.1) and function (4.2) using the equation

$$A_{i_1\ldots i_p} = -\sum_{r,s}\sum_{\alpha=1}^{N}\sum_{\beta=1}^{n}\sum\nolimits^{*} \gamma_\beta c_{\alpha i_1\ldots i_r} c_{\alpha i_{r+1}\ldots\beta\ldots i_{2r-1}} a_{\beta i_{2r}\ldots i_{2r+s-1}} \tag{4.5}$$

Here $c_{\alpha i_1\ldots i_r}$ is a coefficient of the function (4.2); $c_{\alpha i_{r+1}\ldots\beta\ldots i_{2r-1}}$ is the coefficient of the term in the sum of the forms of degree $r\,(r=k,k+1,\ldots,m)$ of the function (4.2), in which the coordinate x_β is constained; γ_β is the degree of the coordinate x_β in this term, equal to the number of repetitions of the index $i_\gamma=\beta\ (r+1\leqslant\gamma\leqslant 2r-1)$ in the coefficient $c_{\alpha i_{r+1}\ldots\beta\ldots i_{2r-1}}$; $a_{\beta i_{2r}\ldots i_{2r+s-1}}$ is the coefficient of system (4.1); $\sum\limits_{r,s}$ is the symbol of summation with respect to all the values r, s, for which $2r+s-1=p$; Σ^* is the symbol of summation over the permutations of the integral values of the indices $i_1,\ldots,i_p=1,\ldots,n$ while preserving the condition: $i_1\leqslant\ldots\leqslant i_r,\ i_{r+1}\leqslant\ldots\leqslant i_{2r-1},\ i_{2r}\leqslant\ldots\leqslant i_{2r+s-1}$ $(r=k,\ k+1,\ \ldots,\ m;\ s=2h-1,\ 2h,\ \ldots,\ 2l-1;\ p=2r+s-1)$.

We shall obtain the conditions of positive definiteness of the function dv/dt (4.4), using Theorem 1. We shall introduce the mapping

$$y_1=x_1^{l+m-1},\quad y_2=x_1^{l+m-2}x_2,\ldots,\quad y_j=x_{i_1}x_{i_2}\ldots x_{i_q},\quad y_{N_0}=x_n^{h+k-1} \tag{4.6}$$
$$j=1,\ldots,N_0;\ i_1,\ i_2,\ \ldots,\ i_q=1,\ldots,n;\ l+m-1\geqslant q\geqslant h+k-1$$

i.e. between the index j and the group index $i_1i_2,\ldots i_q$ there is a one-to-one correspondence. According to Eq.(1.5)

$$N_0=\sum_{q=h+k-1}^{l+m-1}N_q,\quad N_q=C_{n+q-1}^{q} \tag{4.7}$$

We shall write the quadratic form in the space R^{N_0} of the vectors $y=(y_1,\ldots,y_{N_0})$

$$P_0(y)=\sum_{j_1=1}^{N_0}\sum_{j_2=1}^{N_0}B_{j_1j_2}y_{j_1}y_{j_2},\quad B_{j_1j_2}=B_{j_2j_1} \tag{4.8}$$

the coefficients of which are connected with the coefficients of the function dv/dt (4.4) using equations similar to Eq.(1.9)

$$A_{i_1\ldots i_p}=(2-\delta_{j_1j_2})\sum_{q=h+k-1}^{l+m-1}\sum\nolimits^{*}B_{j_1j_2}= \tag{4.9}$$
$$(2-\delta_{j_1j_2})\sum_{q=h+k-1}^{l+m-1}\sum\nolimits^{*}B_{i_1\ldots i_q,\,i_{q+1}\ldots i_p}$$
$$j_1,j_2=1,\ldots,N_0;\quad i_1,\ldots,i_j=1,\ldots,n;$$
$$p=2(h+k-1),\ldots,2(l+m-1)$$

where Σ^* is the symbol of summation over the permuations of the integral values of the indices $i_1,\ldots,i_p$ for which the following conditions hold: $i_1\leqslant\ldots\leqslant i_q,\ i_{q+1}\leqslant\ldots\leqslant i_p\ (j_1\rightleftarrows i_1\ldots i_q,\ j_2\rightleftarrows i_{q+1}\ldots i_p)$.

According to Thoerem 1, for the positive definiteness of the sum of the forms dv/dt (4.4) it is sufficient that the quadratic form $P_0(y)$ (4.8) is non-negative with respect to N_0 variables $y_1,\ldots,y_{N_0}$ and positive definite with respect to N_q variables $y_{N_0-N_q+1},\ldots,y_{N_0}$ (4.6), to which corresponds the form of degree $2q$ in the function dv/dt (4.4), where q takes some value from the set $\{h+k-1,\ h+k,\ \ldots,\ l+m-1\}$.

Suppose $B_{j_1j_2}(a_{\beta i_1\ldots i_s}, c_{\alpha i_1\ldots i_r})$ are the coefficients of the quadratic form (4.8), determined from Eqs.(4.5) and (4.9); N_q is the number of variables $y_{N_0-N_q+1},\ldots,y_{N_0}$ in the quadratic form $P_0(y)$ (4.8); the numbers N,N_0,N_q are calculated using Eqs.(1.5), (4.7); h,l are positive integers, determined using the specified system (4.1). Then, applying to the quadratic form $P_0(y)$ (4.8) the sign determinancy criterion proved in Theorem 2, we arrive at the following statement.

Theorem 3. For the global asymptotic stability of the zero solution of the set of ordinary differential Eq.(4.1) it is sufficient that we have the elements $c_{\alpha i_1\ldots i_r}$ $(\alpha=1,\ldots,N;\ i_1,\ldots,i_r=1,\ldots,n;\ k\leqslant r\leqslant m)$ of the real non-singular $(N\times N)$-matrix (4.3) and the real numbers b_{ij}, which are determined by the coefficients $B_{j_1j_2}$ using the recurrence formula

$$b_{ij}=\frac{1}{b_{ii}}\Big[B_{ij}(a_{\beta i_1\ldots i_s}, c_{\alpha i_1\ldots i_r})-\sum_{k_*=1}^{i-1}b_{ki}b_{kj}\Big] \tag{4.10}$$
$$i=1,\ldots,N_0;\ j=i,\ i+1,\ \ldots,\ N_0;\ 1\leqslant k<i$$

under condition

$$b_{ii}\neq 0,\ \forall i=N_0-N_q+1,\ N_0-N_q+2,\ \ldots,\ N_0 \tag{4.11}$$

where $a_{\beta i_1 \ldots i_s}$ $(\beta = 1, \ldots, n;\ i_1, \ldots, i_s = 1, \ldots, n;\ s = 2h-1, 2h, \ldots, 2l-1)$ are the coefficients of system (4.1).

Proof. Suppose the real numbers shown in Theorem 3 exist. Then the total derivative dv/dt (4.4) of the negative definite function (4.2) with respect to time t by virtue of system (4.1) is a sum of forms, bordered by forms of even degree. According to Theorem 1, for the positive definiteness of the function dv/dt (4.4) it is sufficient that the quadratic form $P_0(y)$ (4.8) be non-negative with respect to N_0 variables $y_1, \ldots, y_{N_0}$ and positive definite with respect to N_q variables $y_{N_0-N_q+1}, \ldots, y_{N_0}$. Theorem 2 gives the necessary and sufficient condition of non-negativity of $P_0(y)$ (4.8) with respect to N_0 variables $y_1, \ldots, y_{N_0}$ and positive definiteness equally with respect to N_q variables $y_{N_0-N_q+1, \ldots, y_{N_0}}$.

Note that the form of degree $2q$, which will guarantee the positive definiteness of the function dv/dt (4.4), is unknown in advance. The number q is thereby unknown. Therefore we will give values from 1 to N_0 to the index i in Eq. (4.10) instead of the values $1, N_0 - N_q + 1, \ldots, N_0$ determined in Theorem 2, i.e. in the proof of the theorem the class of functions (4.4) from which the positive definite sum of the forms is sought is extended.

Thus, the existence of the numbers b_{ij} (4.10) under condition (4.11) is a sufficient condition for the positive definiteness of the function dv/dt (4.4), and this means the sufficient condition of global asymptotic stability of the zero solution of system (4.1) /6/. The theorem is proved.

Note. In the proof of Theorem 3 the numbers k and $m\ (k \leqslant m)$ are regarded as fixed. In general, they are chosen from the natural series, which produce a wide class of Lyapunov functions (4.2).

Corollary. For the asymptotic stability of the zero solution of a set of linear differentia equations with constant coefficients it is necessary and sufficient that the conditions of Theorem 3 hold with

$$s = h = l = 1,\ r = k = m = 1,\ N = N_0 = N_q = n,\ q = 1,\ p = 2 \tag{4.12}$$

Proof. Necessity. Suppose system (4.1) when $s = h = l = 1$ has an asymptotically stable zero solution. Then all the roots of the characteristic equation have negative real parts and according to Lyapunov's thoerem /6/, whatever the previously specified positive definite quadratic form $X^{(2)}(x, A_{i_1 i_2})$ there exists one and only one negative definite quadratic form (4.2), which satisfies Eq. (4.4) when $p = 2$. The existence of the $(N \times N)$-matrix (4.3) when $N = n$ and the numbers b_{ij} (4.10), (4.11) now follows from the criterion of sign determinancy of the quadratic form in /3/.

Sufficiency. The proof of the sufficiency of the corollary is similar to the proof of Theorem 3 under condition (4.12).

An analysis of the stability of the zero solution of system (4.1) on the basis of Theorem 3 can be carried out in the following order.

1°. From the specified system (4.1) the powers of $s = 2h-1, 2h, \ldots, 2l-1$ polylinear froms, on the right-hand side of this system, and the number n, are determined. Hence we obtain the numbers h, l.

2°. The values $k, m\ (k < m)$ are chosen from the natural series of positive integers.

3°. N is calculated using Eq. (1.5).

4°. The upper triangular non-singular $(N \times N)$-matrix (4.3) of the real numbers $c_{\alpha i_1 \ldots i_r}$, $\alpha = 1, \ldots, N;\ i_1, \ldots, i_r = 1, \ldots, n;\ r = k, k+1, \ldots, m$ is specified arbitrarily.

5°. N_0 and N_q are calculated using Eq. (4.7) for all q.

6°. All the coefficients $A_{i_1 \ldots i_p}$ are determined using Eq. (4.5).

7°. The coefficients $B_{j_1 j_2}\ (j_1, j_2 = 1, \ldots, N_0)$ are determined from the set of linear Eq. (4.9).

8°. The real numbers b_{ij} are determined using Eq. (4.10) and condition (4.11) is verified.

9°. If all the numbers b_{ij} of Step 8 exist and satisfy condition (4.11), a conclusion is drawn: the zero solution of the specified system (4.1) is globally asymptotically stable.

Otherwise the chosen Lyapunov function $v(x)$ (4.2) does not enable us to establish the stability of the motion and we should return to Step 2, choose other values k, m and repeat the calculation process. Note that the element of arbitrariness is also contained in Steps 4 and 7, to which we sould also return as necessary.

Example. We shall examine the stability of the longitudinal motion of an aircraft bearing in mind the non-linearity of aerodynamic coefficients and non-linear connections between the angle of attack $\alpha = x_1$ and the pitch velocity $\omega_z = x_2$. We shall consider the equations of the perturbed motion in the form /7/

$$\begin{aligned} dx_\beta/dt = {} & a_{\beta 1}x_1 + a_{\beta 2}x_2 + a_{\beta 11}x_1^2 + a_{\beta 12}x_1x_2 + a_{\beta 22}x_2^2 + \\ & a_{\beta 111}x_1^3 + a_{\beta 112}x_1^2x_2 + a_{\beta 122}x_1x_2^2 + a_{\beta 222}x_2^3, \quad \beta = 1, 2 \end{aligned} \tag{4.13}$$

We shall obtain the global conditions of asymptotic stability of the zero solution of system (4.13) for constant values of the coefficients.

Following the proposed order of the investigation, we shall obtain $n = 2,\ h = 1,\ l = 2,\ s = 1, 2, 3$. We shall specify $k = m = 1$, which corresponds to the quadratic form (4.2). Since $r = k = 1$, then $N = C_{n+r-1}^r = 2$. The $(N \times N)$-matrix (4.3) has the form

$$\begin{Vmatrix} c_{11} & c_{12} \\ 0 & c_{22} \end{Vmatrix}$$

We shall calculate N_q when $q = 1, 2$. We shall obtain $N_1 = 2, N_2 = 3$, then $N_0 = 5$. Using Eq. (4.5) when $p = 2, 3, 4$ and $\gamma_\beta = 1$ we will obtain coefficients of the function of the form (2.1)

$$\begin{gathered} A_{1111} = -c_{11}^2 a_{1111} - c_{11}c_{12}a_{2111},\ A_{1112} = -c_{11}^2 a_{1112} - c_{11}c_{12}a_{1111} - \\ c_{11}c_{12}a_{2112} - (c_{12}^2 + c_{22}^2)\, a_{2111},\ \ldots,\ A_{111} = -c_{11}^2 a_{111} - c_{11}c_{12}a_{211},\ A_1 \\ = -c_{11}^2 a_{112} - c_{11}c_{12}a_{111} - c_{11}c_{12}a_{212} - (c_{12}^2 + c_{22}^2)\, a_{211},\ \ldots,\ A_{22} = \\ -c_{11}c_{12}a_{12} - (c_{12}^2 + c_{22}^2)\, a_{22} \end{gathered} \tag{4.14}$$

Using Eqs.(4.9) we will obtain the coefficients $B_{j_1j_2}\ (j_1, j_2 = 1, \ldots, 5)$. These equations are solved in Sect.2. The solutions of $B_{j_1j_2}$ have the form (2.4) and are expressed by the coefficients (4.14).

Using Eqs.(4.10) we will obtain the numbers b_{ij}:

$$\begin{gathered} b_{11} = \pm B_{11}^{1/2},\quad b_{12} = \frac{B_{12}}{b_{11}},\quad b_{13} = \frac{B_{13}}{b_{11}},\quad b_{14} = \frac{B_{14}}{b_{11}},\quad b_{15} = \frac{B_{15}}{b_{11}} \\ b_{22} = \pm (B_{22} - b_{12}^2)^{1/2},\quad b_{23} = \frac{1}{b_{22}}(B_{23} - b_{12}b_{13}),\quad b_{24} = \frac{1}{b_{22}}(B_{24} - b_{12}b_{14}) \\ b_{25} = \frac{1}{b_{22}}(B_{25} - b_{12}b_{15}),\quad b_{33} = \pm (B_{33} - b_{13}^2 - b_{23}^2)^{1/2} \\ b_{34} = \frac{1}{b_{33}}(B_{34} - b_{13}b_{14} - b_{23}b_{24}),\quad b_{35} = \frac{1}{b_{33}}(B_{35} - b_{13}b_{15} - b_{23}b_{25}) \\ b_{44} = \pm (B_{44} - b_{14}^2 - b_{24}^2 - b_{34}^2)^{1/2} \\ b_{45} = \frac{1}{b_{44}}(B_{45} - b_{14}b_{15} - b_{24}b_{25} - b_{34}b_{35}) \\ b_{55} = \pm (B_{55} - b_{15}^2 - b_{25}^2 - b_{35}^2 - b_{45}^2)^{1/2} \end{gathered} \tag{4.15}$$

Now it is necessary to verify condition (4.11). When $N_1 = 2$ we have the order of variables (2.2) and condition (2.9). When $N_2 = 3$ we have the order of variables (2.10) and condition (2.12).

The existence of the numbers (4.15) under condition (2.9) or (2.12) signifies that the zero solution of system (4.13) is globally asymptotically stable.

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